Applications of the theory of β-models

Dedicated to Prof. M. Kondô
for his 60th Birthday

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β-models are models of the second order analysis for which not only the notion of the natural numbers but also the notion of well-orderings of natural numbers are absolute. This family of models was introduced and studied by Mostowski [20; 21]. Shoenfield [27], Gandy [5] and others made contributions to this theory. We shall give in this note two applications of this theory to set theory.

We shall summarize in the first section some notions and results of the theory of β-models [8; 20; 21]. We shall give in the second section some results about Δ_2^1-functions of natural numbers of mine and Tugué [32]. They were obtained independently of Shoenfield [28] and Gandy [4]. We shall give in the third section applications of the theory of β-models.

As was proved by Mostowski [21], there is a $\sum_2^1$-statement which is not stable for the family of β-models. That is to say, there is an effective non-void complementary analytic set which has not a point in a β-model. Shoenfiel [27] proved, on the other hand, that every $\sum_2^1$-statement is absolute for the β-model $\mathfrak{C}$ consisting of constructible functions of natural numbers [7]. It is therefore interesting to find a condition to decide whether every $\sum_2^1$-statement is absolute for a given β-model. We shall show in the first application that results of Mostowski and Shoenfield can be reversed. We have thence a necessary and sufficient condition to decide whether every $\sum_2^1$-statement is absolute for a given β-model. The effective choice principle of Kondô [15; 1] is used in our proof. By using the effective choice principle, we can show that β-models induced by minimal standard models of various theories [19; 26; 3; 5] do not satisfy our condition. The β-model $\mathfrak{C}_0$ consisting of strongly constructible functions [3], for example, does not satisfy our condition. There is, therefore, an effective complementary analytic set, i.e. $\prod_1^1$-set, which has a constructible function but has no strongly constructible function as its element.

We shall prove in the second application the existence of certain non-standard models of ZF set theory. Rosser-Wang [23; cf. 26] gave an extension of set theory whose models are necessarily non-standard.

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If we examine those models, we see they have non-standard integers. That is to say, they are $\aleph_0$-non-standard models. Mostowski showed the existence of non-β-models and Vopěnka [33] constructed $\aleph_0$-standard non-standard models of set theory. We shall generalize Mostowski's theorem and prove the existence of $\aleph_0$-standard $\aleph_1$-non-standard models of ZF set theory. The existence of such models may give additional security in the use of Cohen's method of forcing.

1. β-models.

We shall summarize in this section some notions and results of the theory of β-models from Grzegorczyk, Mostowski and Ryll-Nardzewski [8] and Mostowski [20; 21].

The *primitive symbols* of the system (A) of analysis consists of the following six groups: (1) Symbols of the propositional calculus $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$. (2) Symbols of the second order predicate calculus $\bigvee$, $\bigwedge$, $=$, ι. (3) The number variables x_i for $i \in N$ where N is the set of natural numbers. (4) The function variables α^i_j for $i \in N - \{0\}$ and $j \in N$. (5) Arithmetical symbols $0, 1, +, \times$. (6) Improper symbols (,).

The set of *terms* and the set of *formulas* are defined in the usual way. Terms and formulas are (well-formed) expressions of the system (A). Superscripts and sufixes are sometimes omitted.

We shall use as metamathematical variables $x, y, x_1, \cdots$ for number variables, $\alpha^i_j, \alpha^i, \alpha_j, \alpha, \beta, \cdots$ for function variables and $\mathfrak{a}, \mathfrak{b}, \cdots$ for sequences of variables. We shall use also $s, t, \cdots$ for terms, $\Phi, \Psi, \cdots$ for formulas and $\Gamma, \Delta, \cdots$ for expressions. The result of substituting s for x in $\Gamma(x)$ is denoted by $\Gamma(s)$ where no fusion of variables occurs. Abbreviations are also used. $\bigvee !x\Phi(x)$ is, for example, the abbreviation of the formula $\bigvee x\Phi(x) \wedge \bigwedge x \bigwedge y(\Phi(x) \wedge \Phi(y) \Rightarrow x=y)$.

Expressions with no quantification of function variables are called *arithmetical*. Formulas in prenex forms are classified in the usual way.

The *axioms of the system* (A) consists of the following six groups (cf. [9; 11]):

(1) Axioms of the propositional calculus.
(2) Axioms of the second order predicate calculus.
(3) Axioms for identity.
(4) Church's axiom

$$(\bigvee !x\Phi(x) \wedge \Phi(\iota x\Phi(x))) \vee (\neg \bigvee !x\Phi(x) \wedge \iota x\Phi(x)=0) .$$

(5) Axioms of Peano's arithmetic.
(6) Leśniewski's axiom

$$\bigvee \alpha^i \bigwedge x_i \cdots \bigwedge x_i(\alpha^i(x_1, \cdots, x_i)=s) ,$$

where the term s does not contain α^i free.

The *rules of inference* of the system (A) are modus ponens and rules of inference of the second order predicate calculus.

A formula is *provable* if it can be deduced from axioms by applications of rules of inference. The set of provable formulas are denoted by $\mathrm{Cn}(A)$ or simply by A.

A *frame* $\mathfrak{F}$ for the system (A) is a sequence

$$\langle N_{\mathfrak{F}}, 0_{\mathfrak{F}}, 1_{\mathfrak{F}}, +_{\mathfrak{F}}, \times_{\mathfrak{F}}, F_{\mathfrak{F}}^1, \cdots, F_{\mathfrak{F}}^i, \cdots\rangle$$

for which the following conditions are satisfied:
(1) $0_{\mathfrak{F}}, 1_{\mathfrak{F}} \in N_{\mathfrak{F}}$.
(2) $+_{\mathfrak{F}}, \times_{\mathfrak{F}} \subseteq N_{\mathfrak{F}}^{N_{\mathfrak{F}}^2}$.
(3) $F_{\mathfrak{F}}^i \subseteq N_{\mathfrak{F}}^{N_{\mathfrak{F}}^i}$.
The sub-sequence $\langle N_{\mathfrak{F}}, 0_{\mathfrak{F}}, 1_{\mathfrak{F}}, +_{\mathfrak{F}}, \times_{\mathfrak{F}}\rangle$ is called the *arithmetical part* of the frame $\mathfrak{F}$ and the sub-sequence $(F_{\mathfrak{F}}^1, \cdots, F_{\mathfrak{F}}^i, \cdots)$ is called the *analytical part* of the frame $\mathfrak{F}$.

We shall say a frame $\mathfrak{F}$ is *denumerable* if all sets $F_{\mathfrak{F}}^i$ are denumerable. We shall say a frame $\mathfrak{F}$ is a *sub-frame* of $\mathfrak{G}$, in symbol $\mathfrak{F} \subseteq \mathfrak{G}$, if the two arithmetical parts are identical and $F_{\mathfrak{F}}^i \subseteq F_{\mathfrak{G}}^i$ for all $i \in N - \{0\}$.

Let $\langle N, 0, 1, +, \times\rangle$ be the structure of natural numbers. The frame $\mathfrak{P} = \langle N, 0, 1, +, \times, N^{N^1}, \cdots, N^{N^i}, \cdots\rangle$ is called the *principal* frame of the system (A). A frame $\mathfrak{F}$ is called an *ω-standard frame* or an *ω-frame* if the arithmetical part of the frame $\mathfrak{F}$ is isomorphic to that of the principal frame. Let ν be an isomorphism of the two parts. Such an isomorphism is evidently unique. The isomorphism ν induces an injection of the analytical part of the frame $\mathfrak{F}$ into the analytical part of the principal frame $\mathfrak{P}$. That is, $\nu(\varphi)(a_1, \cdots, a_i) = \nu(\varphi(\nu^{-1}(a_1), \cdots, \nu^{-1}(a_i)))$, for $\varphi \in F_{\mathfrak{F}}^i$. The ω-frame $\mathfrak{F}$ is a sub-frame of the principal frame by this identification and we have to consider only the analytical part of the ω-frame $\mathfrak{F}$.

A *valuation* or *assignment* (with respect to a frame $\mathfrak{F}$) is a mapping defined on the set of variables for which the following conditions are satisfied: (1) $f(x_i) \in N_{\mathfrak{F}}$, (2) $f(\alpha_j^i) \in F_{\mathfrak{F}}^i$.

We can define, by mathematical induction, the value $\mathrm{Val}_{\mathfrak{F},f}(\Gamma)$ of an expression Γ for the valuation f (with respect to the frame $\mathfrak{F}$). All symbols are interpreted naturally. $\mathrm{Val}_{\mathfrak{F},f}(\Phi)$ is 0 or 1 and $\mathrm{Val}_{\mathfrak{F},f}(s)$ is in the set $N_{\mathfrak{F}}$. $\mathrm{Val}_{\mathfrak{F},f}(\Gamma)$ coincides with $\mathrm{Val}_{\mathfrak{F},g}(\Gamma)$ if the two valuations f and g are identical on the (finite) set of free variables of the expression Γ. We can therefore define $\mathrm{Val}_{\mathfrak{F},h}(\Gamma)$ for a mapping h which can be extended to a valuation and is defined at least on the set of free variables of the expression Γ. We shall use $f, g, h, \cdots$ for such mappings. We shall use abbreviations $\mathrm{Val}_{\mathfrak{F}}\,\Gamma[a, b]$, $\mathrm{Val}_{\mathfrak{F}}\,\Gamma([a], [b])$ etc. for $\mathrm{Val}_{\mathfrak{F},h}(\Gamma(x_i, x_j))$ where $h(x_i) = a$ and $h(x_j) = b$.

We write $\models_{\mathfrak{F}} \Phi[f]$ for $\mathrm{Val}_{\mathfrak{F},f}(\Phi) = 0$ and say Φ is satisfied by the valuation f (in the frame $\mathfrak{F}$). If the formula Φ is satisfied by all valuations in the frame $\mathfrak{F}$, then we say the formula Φ is true in the frame $\mathfrak{F}$. If every axioms of the system (A) are true in a frame $\mathfrak{F}$, we say $\mathfrak{F}$ is a *model of the system* (A). Provable formulas are true in all models. The principal frame is evidently a model and is called the *principal model* of the system (A). An *ω-model* is a model which are ω-standard.

Let $\mathfrak{F}$ be a sub-frame of the frame $\mathfrak{G}$. We say, following Gödel [7], an expression Γ is *absolute* with respect to $\mathfrak{F}$ and $\mathfrak{G}$, if the equation $\mathrm{Val}_{\mathfrak{F},f}(\Gamma)=\mathrm{Val}_{\mathfrak{G},f}(\Gamma)$ holds for every valuation f with respect to $\mathfrak{F}$. If the frame $\mathfrak{G}$ is principal, we say simply the expression Γ is absolute for the frame $\mathfrak{F}$. Let C be a set of sub-frames of a frame $\mathfrak{G}$. We say [16] an expression Γ is *stable* with respect to the family C and $\mathfrak{G}$, if the expression Γ is absolute with respect to every frame in C and $\mathfrak{G}$. If the frame $\mathfrak{G}$ is principal, we say simply the expression Γ is stable for the family C. Arithmetical expressions are evidently stable for the family of ω-frames.

We say a sub-frame $\mathfrak{F}$ of $\mathfrak{G}$ is an *elementary sub-frame* of $\mathfrak{G}$, if every expressions is absolute with respect to $\mathfrak{F}$ and $\mathfrak{G}$. The theorem of Skolem [29, 31] shows: For every frame $\mathfrak{G}$, there is a denumerable elementary sub-frame $\mathfrak{F}$ of $\mathfrak{G}$. If the frame $\mathfrak{G}$ is a model of the system (A), then elementary sub-frames of $\mathfrak{G}$ are also models of the system (A). If we apply this theorem to the principal model $\mathfrak{P}$, we have denumerable ω-models.

Let us denote by $\mathrm{Ord}\,(\alpha^2)$ the formula

$$\begin{aligned}\bigwedge x \bigwedge y \bigwedge z(\{&[\alpha^2(x, z)=0 \vee \alpha^2(z, x)=0]\\ &\wedge[\alpha^2(y, z)=0 \vee \alpha^2(z, y)=0]\\ &\Rightarrow \alpha^2(x, x)=0 \wedge [\alpha^2(x, y)=0 \vee x=y \vee \alpha^2(y, x)=0]\}\\ &\wedge[\alpha^2(x, y)=0 \wedge \alpha^2(y, x)=0 \Rightarrow x=y]\\ &\wedge[\alpha^2(x, y)=0 \wedge \alpha^2(y, z)=0 \Rightarrow \alpha^2(x, z)=0])\,.\end{aligned}$$

$\mathrm{Bord}\,(\alpha^2)$ is the formula

$$\mathrm{Ord}\,(\alpha^2) \wedge \bigwedge \beta^1 \bigvee x\{\neg\, \alpha^2(\beta^1(x+1), \beta^1(x))=0 \vee \beta^1(x+1)=\beta^1(x)\}$$

For any ω-frame $\mathfrak{F}$ and $\varphi \in F^2_{\mathfrak{F}}$, $\models_{\mathfrak{F}} \mathrm{Ord}\,[\varphi]$ is equivalent to the fact that φ is a linear ordering in N. The formula $\mathrm{Ord}\,(\alpha^2)$ is evidently stable for the family of ω-frames. An ω-model $\mathfrak{B}$ is called a *β-model* if the formula $\mathrm{Bord}\,(\alpha^2)$ is absolute for the frame $\mathfrak{B}$.

Remark. The definition of β-models given above is not identical to that of Mostowski [21]. Our definition is, however, equivalent to that of Mostowski, as β-models in his sense are ω-models [21, Lemma

3.5].

Examples of β-models are given by Mostowski [21]. Every elementary sub-frame of a β-model is a β-model. The frames determined by a well-founded model of ZF set theory are also β-models.

Formalizing the theory of sieves in the system (A), we can prove the following fundamental theorem for β-models.

Theorem 1. *Every Π_1^1-formula is stable for the family of the β-models.*

Remark. We can prove this theorem similarily as for the theorem 6.3 of Mostowski [21].

By using Theorem 1, we can prove the following

Corollary. *Let Ψ be a Σ_2^1-statement. If Ψ holds in a β-model $\mathfrak{B}$, then it holds in the principal model $\mathfrak{P}$* [21, Theorem 6.3].

Proof. For some arithmetical formula $\Phi(\alpha, \beta)$, the statement Ψ is logically equivalent to the statement $\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ in the system (A) [21, Section 2]. By our hypothesis, there is a function φ in $\mathfrak{B}$ such that $\models_{\mathfrak{B}}\wedge\alpha\Phi(\alpha, [\varphi])$. By Theorem 1, $\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\varphi])$. As φ is in $\mathfrak{P}$, the statement $\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ holds for $\mathfrak{P}$.

2. Δ_2^1-functions.

A number-theoretic function is a Δ_n^1-function if its representing predicate is a Δ_n^1-predicate [12; 2; 27]. The frame consisting of the Δ_n^1-functions is denoted by $\mathfrak{D}_n$. The frame $\mathfrak{D}_1$ is identical to the frame consisting of the hyper-arithmetical functions [12; 13]. It must be noted that the frame $\mathfrak{D}_1$ does not form a model of analysis [6].

We can define analogous notion for objects of higher types [14]. We shall introduce objects $\mathbf{E}$ and $\mathbf{E}_1$ of type-2 by the following conditions:

$$\mathbf{E}(\varphi)=i \leftrightarrow [(\exists x)(\varphi(x)=0) \ \& \ i=0]\vee[\sim(\exists x)(\varphi(x)=0) \ \& \ i=1] \,,$$

$$\begin{aligned}\mathbf{E}_1(\varphi)=i \leftrightarrow {} & [(\forall\psi)(\exists x)(\varphi(\bar{\psi}(x))=0) \ \& \ i=0] \\ & \wedge[\sim(\forall\psi)(\exists x)(\varphi(\bar{\psi}(x))=0 \ \& \ i=1] \,.\end{aligned}$$

The function $\mathbf{E}$ was introduced by Kleene [14] and the function $\mathbf{E}_1$ was introduced by Tugué [32]. By applying contraction techniques of quantifiers [14], we see the function $\mathbf{E}$ is a $\Delta_2^{0,1}$-function and the function $\mathbf{E}_1$ is a $\Delta_2^{1,1}$-function.

Since the work of Addison [2] has appeared, we know clearly the relation between classical descriptive set theory [18] and recursive function theory [11; 12; 13]. Δ_1^1-functions correspond to Baire's functions and Δ_2^1-functions correspond to B_2-functions. Many mathematicians investigated B_2-functions. Some problems about that family was recognized difficult to solve. Efforts to construct a natural hierarchy for B_2-functions, for example, were in vain [10; 17; 24]. The following negative answer to a problem of Tugué [32] is an analogous phenomenon for Δ_2^1-functions.

Let us denote by $\mathfrak{S}(\mathbf{F})$ the frame consisting of the functions recursive in the object $\mathbf{F}$. As was proved by Kleene [14, Theorem XLVIII], the identity $\mathfrak{S}(\mathbf{E})=\mathfrak{D}_1$ holds. Tugué showed the inclusion $\mathfrak{S}(\mathbf{E}_1)\subseteq\mathfrak{D}_2$ and asked whether the converse inclusion holds. His problem was solved negatively by us and him. It was also solved by Shoenfield [28] and Gandy [4].

Theorem 2.1. *The frame $\mathfrak{S}(\mathbf{E}_1)$ is a proper sub-frame of the frame $\mathfrak{D}_2$.*

Proof. The frame $\mathfrak{S}(\mathbf{E}_1)$ is a sub-frame of the frame $\mathfrak{D}_2$ [32, Lemma 3 or 14, Theorem XLV]. Consider the predicate $\{z\}(a, \mathbf{E}_1)\simeq i$ [14, Section 5]. We can show that it is a $\prod_2^1$-predicate [32, Lemma 2; 14, Theorem XXVIII]. Let us define two sets S_0 and S_1 by the following condition:

$$x\in S_i \leftrightarrow \{(x)_0\}((x)_1, \mathbf{E}_1)\simeq i\ .$$

They are disjoint $\prod_2^1$-sets. By the first separation principle of Novikov [22; 2], they are separable by a Δ_2^1-set T. The representing function τ of T is a Δ_2^1-function. By a diagonal reasoning, we see that the function τ is not in the frame $\mathfrak{S}(\mathbf{E}_1)$. [cf. 21, Theorem 6.9].

Shortly after we proved our Theorem, Tugué found the following generalization of it. It must be noted that his proof requires neither separation nor uniformization principle.

Theorem 2.2. *If $\mathbf{F}$ is a $\Delta_n^{1,1}$-object of type-2, then the frame $\mathfrak{S}(\mathbf{F})$ is a proper sub-frame of the frame $\mathfrak{D}_n$ (for $n\geq 2$).*

Proof. By using the Theorems XXVIII and XLV of Kleene [14], we can show that the predicate $\{z\}[a, \varphi, \mathbf{F}]\simeq w$ is a $\prod_n^{1,1}$-predicate. Consider the predicate $\mathbf{E}(z, a, \varphi, \mathbf{F})$ [14, Section 5]. If we apply the Theorem XLV of Kleene [14] and contraction techniques to that predicate, we see that the predicate $\mathbf{E}(z, a, \varphi, \mathbf{F})$ and thence the predicate $\{z\}[a, \varphi, \mathbf{F}]\simeq w$ are $\sum_n^{1,1}$-predicates. $\{z\}[a, \varphi, \mathbf{F}]\simeq w$ is therefore a Δ_n^1-predicate and the predicate $\{z\}(z, \mathbf{F})\simeq 0$ is a Δ_n^1-predicate. The representing function τ of the predicate $\{z\}(z, \mathbf{F})\simeq 0$ is evidently a Δ_n^1-function and is not in the frame $\mathfrak{S}(\mathbf{F})$.

Remark. If we inspect our proofs of Theorems 2.1 and 2.2, we see that proper inclusions $\mathfrak{S}^1(\mathbf{E}_1)\subset\mathfrak{D}^1_2$ and $\mathfrak{S}^1(\mathbf{F})\subset\mathfrak{D}^1_n$ were at the same time proved, where the superscripts show the highest type of variables allowed.

3. Applications.

As was proved by Mostowski [21, Corollary 6.7], there is a $\sum^1_2$-statement Ψ which is not stable for the family of β-models. We may assume that the statement Ψ is of the form $\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ where $\Phi(\alpha, \beta)$ is an arithmetical formula [21, Lemma 2.16]. By our hypothesis, there is a β-model $\mathfrak{B}$ such that the propositions

$$\models_{\mathfrak{P}}\vee\beta\wedge\alpha\Phi(\alpha, \beta) \quad\text{and}\quad \models_{\mathfrak{B}}\vee\beta\wedge\alpha\Phi(\alpha, \beta)$$

are not equivalent. The proposition $\models_{\mathfrak{P}}\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ is true and the proposition $\models_{\mathfrak{B}}\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ is not true [21, Theorem 6.3]. If we consider the $\prod^1_1$-set $\{\varphi \mid \models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\varphi])\}$, it is not empty but has no function in the β-model $\mathfrak{B}$.

Shoenfield proved, on the other hand, that every $\sum^1_2$-statement is absolute for the β-model $\mathfrak{C}$ consisting of the constructible functions of natural numbers [27, Corollary 1]. It follows from his proof that every Δ^1_2-function is constructible, i.e. $\mathfrak{D}_2\subseteq\mathfrak{C}$. We shall show in the first application that results of Mostowski and Shoenfield can be reversed. We have then a necessary and sufficient condition to decide whether every $\sum^1_2$-statement is absolute for a given β-model.

Theorem 3.1. *Let $\mathfrak{B}$ be a β-model. Every $\sum^1_2$-statement is absolute for the frame $\mathfrak{B}$ if and only if the frame $\mathfrak{D}_2$ is a sub-frame of $\mathfrak{B}$, i.e. $\mathfrak{D}_2\subseteq\mathfrak{B}$.*

Proof. Let us assume that every $\sum^1_2$-statement are absolute for a given β-model $\mathfrak{B}$. Let φ be the unique solution of a $\prod^1_1$-condition $\lambda\psi\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\psi])$ where the formula $\Phi(\alpha, \beta)$ is arithmetical. As φ is a solution of the condition $\lambda\psi\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\psi])$, we have $\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\varphi])$ and hence $\models_{\mathfrak{P}}\vee\beta\wedge\alpha\Phi(\alpha, \beta)$. As every $\sum^1_2$-statement is absolute for the frame $\mathfrak{B}$, the proposition $\models_{\mathfrak{B}}\vee\beta\wedge\alpha\Phi(\alpha, \beta)$ holds. By the definition of truth, there is a function η such that $\models_{\mathfrak{B}}\wedge\alpha\Phi(\alpha, [\eta])$. By Theorem 1, the proposition $\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\eta])$ holds. As φ was the unique solution of the condition $\lambda\psi\models_{\mathfrak{P}}\wedge\alpha\Phi(\alpha, [\psi])$, the function η is identical to the function φ, i.e. $\varphi\in\mathfrak{B}$. As was proved by us [30, Theorem 1], every Δ^1_2-function ξ is hyper-arithmetical in a function φ which is a unique solution of a $\prod^1_1$-condition. There is, therefore, an arithmetical formula $\Psi(\alpha, \beta, x_1, \cdots, x_n, y)$ for which the equivalence

$$\xi(a_1, \cdots, a_n) = b \leftrightarrow \models_{\mathfrak{P}}\wedge\alpha\Psi(\alpha, [\varphi, a_1, \cdots, a_n, b])$$

holds for every $a_1, \cdots, a_n$ and b. By Theorem 1, the proposition $\xi(a_1, \cdots, a_n)=b$ is equivalent to the proposition

$$\models_{\mathfrak{P}} \wedge \alpha \Psi(\alpha, [\varphi, a_1, \cdots, a_n, b]) .$$

Consider the term $\iota y \wedge \alpha \Psi(\alpha, \beta, x_1, \cdots, x_n, y)$. By the axiom of Leśniewski, we see that the function ξ is in the frame $\mathfrak{B}$, i.e. $\mathfrak{D}_2 \subseteq \mathfrak{B}$. Let us assume conversely that the frame $\mathfrak{D}_2$ is a sub-frame of the β-model $\mathfrak{B}$. Consider a $\sum_2^1$-statement $\vee \beta \wedge \alpha \Phi(\alpha, \beta)$. If the statement $\vee \beta \wedge \alpha \Phi(\alpha, \beta)$ holds for the β-model $\mathfrak{B}$, it holds also for the principal frame $\mathfrak{P}$ [21, Theorem 6.3]. Let us assume the statement $\vee \beta \wedge \alpha \Phi(\alpha, \beta)$ to hold for the principal frame $\mathfrak{P}$. By the uniformization principle of Kondô [15; 1; 25], there is a Δ_2^1-function φ such that $\models_{\mathfrak{P}} \wedge \alpha \Phi(\alpha, [\varphi])$. By our hypothesis, the function φ is in the β-model $\mathfrak{B}$. By Theorem 1, $\models_{\mathfrak{B}} \wedge \alpha \Phi(\alpha, [\varphi])$ and thence $\models_{\mathfrak{B}} \vee \beta \wedge \alpha \Phi(\alpha, \beta)$.

Remark. It must be noted that the frame $\mathfrak{D}_2$ does not form a model for analysis [20, p. 404]. We can therefore replace the inclusion $\mathfrak{D}_2 \subseteq \mathfrak{B}$ by the proper inclusion $\mathfrak{D}_2 \subset \mathfrak{B}$ in our Theorem 3.1.

By using effective choice principle of Kondô, we see that β-models induced from minimal standard models do not satisfy our condition and consequently that all $\sum_2^1$-statements are not absolute for those β-models.

Corollary 1. *Let $\mathfrak{B}_0$ be the minimal β-model* [5]. *All $\sum_2^1$-statement are not absolute for $\mathfrak{B}_0$.*

Proof. By our Theorem 3.1 and Corollary 6.7 of Mostowski [21], we see that there is a β-model $\mathfrak{B}$ which does not contain all Δ_2^1-functions. As the frame $\mathfrak{B}_0$ is a sub-frame of $\mathfrak{B}$, $\mathfrak{B}_0$ does not satisfy our condition, too.

Let $\mathfrak{C}_0$ be the β-model consisting of the strongly constructible functions of natural numbers [3]. We can prove a similar proposition for the frame $\mathfrak{C}_0$. We shall recall semantics for *ZF* set theory.

Semantical notions for the system (A) of analysis was discussed in the first section. We can do the same for the system *ZF* of set theory. A *frame for ZF set theory* is a sequence $\langle M, R \rangle$ where M is a non-empty set and R is a binary relation in M. Notions, such as valuation, satisfaction and truth, are defined similarly. Consider frames

$$\mathfrak{N}_\mu = \langle N, \{(a, b) \mid \mu(a, b)=0\} \rangle$$

for $\mu \in N^{N^2}$. As was proved by Mostowski [21, Section 4],

Lemma 1. *The predicate $\models_{\mathfrak{N}_\mu} \Phi[\varphi]$ is hyper-arithmetical.*

We say a model $\mathfrak{M}$ for set theory is *standard* if there is no infinite descending chain in $\mathfrak{M}$. We say a model $\mathfrak{M}$ for set theory is $\aleph_i$*-standard* if there is no infinite descending chain below $\aleph_i$ of the model $\mathfrak{M}$. The standard frame $\mathfrak{M}$ induces naturally a frame $\mathfrak{A}(\mathfrak{M})$ for analysis and that frame $\mathfrak{A}(\mathfrak{M})$ is a model for the system (A) [21, Section 7]. We can generalize this for $\aleph_0$-standard models for set theory. Let ν be the canonical isomorphism of natural numbers onto the natural numbers of the $\aleph_0$-standard model $\mathfrak{M}$. An element a of the model is called a *k-ary function* of natural numbers if we have the proposition $\models_{\mathfrak{M}}\Psi[a]$ where $\Psi(x)$ is the formula $x \operatorname{Func} N^k \wedge W(x) \subseteq N$. For such an element a, we can define a k-ary function $j(a)$ of natural numbers by the following condition:

$$j(a)(b_1, \cdots, b_k) = c \leftrightarrow \models_{\mathfrak{M}} (x^{\text{c}}(y_1, \cdots, y_k) = z)[a, \nu(b_1), \cdots, \nu(b_k), \nu(c)] \; .$$

Those functions constitute an ω-frame $\mathfrak{A}(\mathfrak{M}) = \langle F^1_{\mathfrak{M}}, \cdots, F^k_{\mathfrak{M}}, \cdots \rangle$. We have the following

Lemma 2. *If $\mathfrak{M}$ is an $\aleph_0$-standard model for ZF set theory, the frame $\mathfrak{A}(\mathfrak{M})$ is a model of the system (A).*

Proof. Similarly as in Mostowski [21, Lemma 7.15].

We can prove now the

Corollary 2. *Let $\mathfrak{C}_0$ be the β-model consisting of the strongly constructible functions of natural numbers. All $\sum^1_2$-statement are not absolute for the model $\mathfrak{C}_0$ if there is a strongly inaccessible cardinal.*

Proof. By our hypothesis, there is a standard model for set theory [19]. By Skolem's argument, there is a denumerable standard model. It is isomorphic to a frame $\mathfrak{N}_\mu$. By Lemma 1, the condition "$\mathfrak{N}_\mu$ is a standard model" $C_\beta(\mu)$ is a $\prod^1_1$-condition. As we have proved, the set $\hat{\mu}C_\beta(\mu)$ is not empty. By the effective choice principle of Kondô [15; 1], there is a Δ^1_2-function μ in the set $\hat{\mu}C_\beta(\mu)$. The frame $\mathfrak{N}_\mu$ is a standard model and is isomorphic to an ε-model $\mathfrak{M}$ [19]. Consider the canonical isomorphism ι of the natural numbers onto the natural numbers of the model $\mathfrak{N}_\mu$. The function ι and its inverse are hyper-arithmetical in μ by our Lemma 1. For any k-ary function a of natural numbers in the model $\mathfrak{N}_\mu$, the function $j(a)$ is hyper-arithmetical in μ because of the same reason. The frame $\mathfrak{A}(\mathfrak{N}_\mu)$ is therefore a proper sub-frame of $\mathfrak{D}_2$. The frame $\mathfrak{A}(\mathfrak{N}_\mu)$ is evidently identical to the frame $\mathfrak{A}(\mathfrak{M})$. Because of minimality, the frame $\mathfrak{C}_0$ is a sub-frame of the frame $\mathfrak{A}(\mathfrak{M})$. The condition of our Theorem 3.1 is not satisfied for the frame $\mathfrak{C}_0$.

We shall prove the existence of $\aleph_0$-standard, $\aleph_1$-non-standard model for *ZF* set theory [23; 26]. This can be done by generalizing results of Mostowski [21, Corollary 5.12]. By Lemma 1, the condition "$\mathfrak{N}_\mu$

is an $\aleph_0$-standard model" $C_{\aleph_0}(\mu)$ is a $\sum_1^1$-condition and the condition "$\mathfrak{N}_\mu$ is an $\aleph_1$-standard model" $C_{\aleph_1}(\mu)$ is a $\prod_1^1$-condition. The set $ZF_{\aleph_0}$ consisting of the statements true in every $\aleph_0$-standard model is consequently a $\prod_1^1$-set and the set $ZF_{\aleph_1}$ consisting of the sentences true in every $\aleph_1$-standard model is a $\prod_2^1$-set. As was proved by Mostowski for standard models,

Lemma 3. *If $\mathfrak{M}$ is an $\aleph_1$-standard model for ZF set theory, the frame $\mathfrak{A}(\mathfrak{M})$ is a β-model for analysis (A).*

Proof. The frame $\mathfrak{A}(\mathfrak{M})$ is a model for the analysis (A) by our Lemma 2. We shall prove that $\mathfrak{A}(\mathfrak{M})$ is a β-model. Suppose that the model $\mathfrak{A}(\mathfrak{M})$ is not a β-model. There is an element φ such that $\models_{\mathfrak{A}(\mathfrak{M})} \mathrm{Bord}[\varphi]$ and $\sim \models_{\mathfrak{P}} \mathrm{Bord}[\varphi]$. By our definition of the frame $\mathfrak{A}(\mathfrak{M})$, φ is identical to $j(a)$ for some a. The element a defines a well-ordering of natural numbers in the model $\mathfrak{M}$ and that ordering is isomorphic to an ordinal number α_a which is smaller than $\aleph_1$ of the model $\mathfrak{M}$. By our choice of φ, there is an infinite descending chain for the ordinal number α_a. It contradicts $\aleph_1$-standardness of the model $\mathfrak{M}$.

We can now prove the following

Theorem 3.2. *If there is a strongly inaccessible cardinal, there is an $\aleph_0$-standard, $\aleph_1$-non-standard model for ZF set theory.*

Proof. Every formula of analysis Φ can be interpreted as a formula Φ^* of set theory [21, Section 7]. The interpretation $\Phi \leftrightarrow \Phi^*$ is recursive. For every statement Φ, we have the equivalence $\models_{\mathfrak{M}} \Phi^* \leftrightarrow \models_{\mathfrak{A}(\mathfrak{M})} \Phi$. Let Ψ be a $\prod_2^1$-statement. If Ψ is true in the principal model $\mathfrak{P}$, then Ψ is true in every β-model $\mathfrak{B}$ [21, Theorem 6.3]. If $\mathfrak{M}$ is an $\aleph_1$-standard model for set theory, we have $\models_{\mathfrak{A}(\mathfrak{M})} \Psi$ by Lemma 3 and consequently we have $\models_{\mathfrak{M}} \Psi^*$. Let us assume conversely that the statement Ψ^* holds for every $\aleph_1$-standard model $\mathfrak{M}$. By our hypothesis, there is a standard model $\mathfrak{N}$ which has all functions of natural numbers as its element [19]. We have, therefore, the identity $\mathfrak{A}(\mathfrak{N}) = \mathfrak{P}$. As the statement Ψ^* was true for the frame $\mathfrak{N}$, the statement Ψ is also true for the frame $\mathfrak{A}(\mathfrak{N}) = \mathfrak{P}$. We have proved thus that the set $ZF_{\aleph_1}$ is complete in the sense of Post for the family of the $\prod_2^1$-sets. The set $ZF_{\aleph_1} - ZF_{\aleph_0}$ is not empty, as $ZF_{\aleph_0}$ is a $\prod_1^1$-set.

Vopěnka [33] constructed an $\aleph_0$-standard, non-standard model for set theory. He uses the existence of a measurable cardinal. We do not know however how to construct our models as in Vopěnka from the existence of a strongly inaccessible cardinal.

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